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Applications of the Mellin–Barnes integral representation

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Abstract. We apply the Mellin–Barnes integral representation to several situations of interest in mathematical physics. At the purely mathematical level, we derive useful asymptotic expansions of different zeta-functions and partition functions. These results are then employed in different topics in quantum field theory, which include the high-temperature expansion of the free energy of a scalar field in ultrastatic curved spacetime, the asymptotics of the p -brane density of states, and an explicit approach to the asymptotics of the determinants that appear in string theory.

1. Introduction

Expansions in terms of asymptotic series constitute a very important tool in various branches of physics and mathematics. For example asymptotic expansions in inverse powers of large masses of the field have always been a usually employed approximation (see, for example, [1, 2] and references therein). Another example is the high-temperature expansion in different contexts [3, 4]. For the numerical analysis of several functions the use of asymptotic expansions has also been shown to be very powerful [5, 6]. Finally, let us mention applications to the theory of partitions [7], where asymptotic series are basic in the proof of the theorem of Meinardus [8, 9], which is fundamental to the calculation of the asymptotic density of p -brane states [10].

In all the mentioned applications, the asymptotic series arises when calculating a functional determinant, which has to be regularized by some method. Usually, this functional determinant is a one-loop approximation to a functional integral resulting from integrating out the quadratic part of the quantum fluctuations around some background fields extremizing the action [11]. The quadratic part of the quantum fluctuations is often described by elliptic operators of the form

$$E = -D^2 + M^2 \quad D_\mu = \partial_\mu - iA_\mu \quad (1.1)$$

with a gauge potential A_μ and some (effective) mass M^2 of the theory.

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One of the possible regularization schemes for the product of eigenvalues is the zeta-function regularization procedure introduced some time ago [12]. In this procedure the one-loop quantum correction Γ to the action is described by

$$\Gamma = \frac{1}{2} \ln \det E = -\frac{1}{2} [\zeta'_E(0) - \ln \mu^2 \zeta_E(0)]$$

being $\zeta_E(s)$ the zeta-function associated with the operator E , that is

$$\zeta_E(s) = \sum_j \lambda_j^{-s}$$

where the λ_j are the eigenvalues of E . Furthermore, μ is a mass scale that needs to be introduced for dimensional reasons. Thus, in using this scheme, the basic mathematical tool is the analysis of zeta-functions associated with (pseudo-) elliptic differential operators and the determination of its properties depending on the problem under consideration.

On the purely mathematical side of the problem a very interesting and self-contained reference is the one by Jorgenson and Lang [13]. In these lecture notes the authors describe how parts of analytic number theory and parts of the spectral theory of certain operators can be merged under a more general analytic theory of regularized products of certain sequences of numbers satisfying a few basic axioms. However, their exposition is kept on a general level, which is not directly applicable to the physical problems. Physicists often have to deal with very specific situations, generally fulfilling these few basic axioms, but for such specific situations it is more useful if results are given in full detail. There is thus a need for a bridge between the purely mathematical part of the problem and its specific applications in physics. This is part of the motivation of the present article, its main emphasis being on the physical applications described below.

In order to keep the paper self-contained, we have decided to refer not only to the associated mathematical literature, but also to present the techniques in the context of physically relevant situations. So, for example, a detailed treatment of Epstein-type zeta-functions is of great interest, because these functions are essential for the computation of effective actions in non-trivial backgrounds that appear in different contexts [14] and for the analysis of the Casimir effect [15].

The most elementary example is that of the Hurwitz zeta-function

$$\zeta_H(s; a) = \sum_{n=0}^{\infty} (n+a)^{-s}$$

which has been treated in detail only recently in [16–18]. Before, the only derivatives of $\zeta_H(s; a)$ available in the usual tables were

$$\begin{aligned} \frac{\partial}{\partial a} \zeta_H(s; a) &= -s \zeta_H(s+1; a) \\ \frac{\partial}{\partial s} \zeta_H(s; a) \Big|_{s=0} &= \ln \Gamma(a) - \frac{1}{2} \ln(2\pi). \end{aligned}$$

The aim in [16] was to obtain an asymptotic expansion of $\partial \zeta_H(s; a) / \partial s$ valid for all negative values of s using as starting point Hermite's integral representation for $\zeta_H(s; a)$. Later on, this result has been rederived using a direct method connected with the calculus of finite differences [17], applicable also to other functions of interest. The result has been extended to $\partial^n \zeta_H(s; a) / \partial s^n$ in [18], where a very useful recurrent formula has been derived, which completely solves the problem of the calculation of any derivative of the Hurwitz zeta-function. In all these references, it has been mentioned that the asymptotic expansion of $\zeta'_H(s; a)$ may also be simply found by differentiation of the asymptotic expansion of

$\zeta_H(s; a)$; however, this procedure is not justified *a priori* (in fact, it is controlled by a Tauberian theorem).

In order to introduce and illustrate a different, powerful technique for the derivation of asymptotic series, which is the Mellin transformation technique (Jorgenson and Lang also call it vertical transform), we will take once more the Hurwitz zeta-function $\zeta_H(s; a)$. It serves as an elementary example in order to introduce the Mellin technique, which, as mentioned, is applicable in a much wider range [13]. We will derive an asymptotic series expansion for $\partial^n \zeta_H(s; a) / \partial s^n(s; a)$, for all values $n \in \mathbb{N}_0$. In particular we will show again that, in order to obtain the asymptotic series of the derivatives of the Hurwitz zeta function, one may simply differentiate the asymptotic expansion (a method that was proven to be valid in this case by applying standard mathematical results for asymptotic expansion, such as Laplace’s method and Watson’s lemma [16, 5]).

After having presented the method for the case of $\zeta_H(s; a)$, we consider other different quantities, explaining briefly at the beginning of each section how these appear in actual physical situations. In section 3 the more complicated example of an Epstein-type zeta-function is presented. Section 4 is devoted to the treatment of sums which most frequently appear in finite-temperature quantum field theory and in the theory of partitions. We use the new technique in order to rederive the high-temperature expansion of a free scalar field in curved spacetime. Furthermore, we outline a generalization of the theorem of Meinardus [8, 9], which enables one to find the asymptotic state density of p -branes [10]. The last application is concerned with properties of some determinants appearing in (super-) string theory. In the conclusions of the paper, the results presented here are briefly summarized.

2. Asymptotic expansion of the Hurwitz zeta-function

The aim of this section is to explain how the approach works in obtaining the asymptotic expansions of a large class of functions [13]. As an example, we choose the very useful case of the Hurwitz zeta-function $\zeta_H(s; a)$, defined by [19]

$$\zeta_H(s; a) = \sum_{n=0}^{\infty} (n + a)^{-s} \quad 0 < a \leq 1 \quad \text{Re } s > 1 \quad (2.1)$$

and derive an asymptotic expansion for large values of a (for previous treatments see [16–18]).

To start with, we rewrite equation (2.1), as usual, in the form

$$\zeta_H(s; a) = a^{-s} + \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} dt t^{s-1} e^{-(n+a)t}.$$

The key idea is to make use of the complex integral representation of the exponential in the form of an integral of Mellin–Barnes type:

$$e^{-v} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) v^{-\alpha} \quad (2.2)$$

with $\text{Re } v > 0$ and $c \in \mathbb{R}, c > 0$. Restricting it to the part e^{-nt} only, it leads to

$$\begin{aligned} \zeta_H(s; a) &= a^{-s} + \frac{1}{2\pi i \Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} dt t^{s-1} e^{-at} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) n^{-\alpha} t^{-\alpha} \\ &= a^{-s} + \frac{1}{2\pi i \Gamma(s)} \sum_{n=1}^{\infty} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) \Gamma(s - \alpha) n^{-\alpha} a^{\alpha-s}. \end{aligned} \quad (2.3)$$

Now we would like to interchange the summation and integration in order to arrive at an expression in terms of Riemann zeta-functions. By choosing $c > 1$, the resulting sum is absolutely convergent, leading to

$$\zeta_H(s; a) = a^{-s} + \frac{1}{2\pi i \Gamma(s)} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) \Gamma(s - \alpha) \zeta_R(\alpha) a^{\alpha-s}. \tag{2.4}$$

The integrand has poles on the left of the contour at (choosing $s > c$) $\alpha = 1$ (the pole of $\zeta_R(\alpha)$) and $\alpha = -k$, $k \in \mathbb{N}_0$ (the poles of $\Gamma(\alpha)$). All poles are of order one and, with $\zeta(1 - 2m) = -B_{2m}/(2m)$, for $m \in \mathbb{N}$, one easily finds the asymptotic behaviour

$$\zeta_H(s; a) \sim \frac{1}{2} a^{-s} + \frac{1}{s-1} a^{-s+1} + \sum_{k=2}^{\infty} \frac{(s)_{k-1} B_k}{k!} a^{-s-k+1} \tag{2.5}$$

where $(s)_k$ is the Pochhammer symbol $(s)_k = \Gamma(s+k)/\Gamma(s)$. This result agrees (of course) with the known result. However, here it has been rederived with nearly no calculational effort.

Let us now concentrate on the asymptotic expansion of $(\partial^n/\partial s^n)\zeta_H(s; a)$. It has already been proven in the literature [16–18] that the asymptotic expansion of the first derivative of the Hurwitz zeta-function is simply the term by term derivative of the asymptotic expansion (2.5), a procedure which is not justified *a priori* and needs a lengthy demonstration in terms of Watson’s lemma. We would now like to show that, in the case considered above, our procedure is true for *all* derivatives of $\zeta_H(s; a)$.

The proof is the following. Looking at (2.4), it is easily seen that integration and differentiation may safely be commuted. The reason for this is that differentiation does not destroy the rapid decay of the gamma function, which is seen using the representation 8.341.1 in [20] for $\ln \Gamma(z)$. In fact, by doing so no additional poles are created and the residue of the pole is just the derivative of the old one. Thus the asymptotic expansion of $(\partial^n/\partial s^n)\zeta_H(s; a)$ is simply the term by term differentiate of equation (2.5). Formulae for very quick explicit derivation of those (together with some basic examples) can be found in [18].

3. Asymptotic series expansion of Epstein-type zeta-functions

As the next example, we would like to consider the Epstein-type zeta-function

$$E^2(s) = \sum_{n=0}^{\infty} [(n+a)^2 + M^2]^{-s}. \tag{3.1}$$

The range of summation is chosen in such a way that $M = 0$ corresponds to the Hurwitz zeta-function. As the presented calculations will show, other index ranges (depending on the details of the subsequently described physical situations) do not give rise to additional problems and may be treated in exactly the same way.

This type of function, or multi-dimensional generalizations of it, is of importance for different problems in quantum field theory. For example it naturally appears in the context of gauge field mass generation in partially compactified spacetimes of the type $T^N \times \mathbb{R}^n$. In some detail, a massive complex scalar field ϕ defined on $T^N \times \mathbb{R}^n$ with, for example, periodic boundary conditions for each of the toroidal components is coupled to a constant Abelian gauge potential A_μ . Due to the non-trivial topology, constant values of the toroidal components are physical parameters of the theory and the effective potential of the gauge theory will depend on these parameters. For the calculation of the effective potential a detailed knowledge of $E^2(s)$ is necessary, where the parameter a corresponds to an Abelian

gauge potential in a toroidal dimension [21, 22]. The parameter M^2 here plays the role of the mass squared m^2 of the scalar field. Realizing that an imaginary constant gauge potential is equivalent to a chemical potential [4], the relevance of $E^2(s)$ for finite-temperature quantum field theory in Minkowski spacetime and for the phenomenon of Bose–Einstein condensation is also obvious [3]. The case $a = 0$ is relevant to describe topics like topological symmetry breaking or restoration in self-interacting $\lambda\phi^4$ scalar field theories on the spacetime $\mathbb{R}^3 \times S^1$ [14]. There, the effective mass naturally appearing in the theory is $M^2 = m^2 + (\lambda/2)\hat{\phi}^2$, with the classical scalar background field $\hat{\phi}$. Finally, let us mention the appearance of similar functions in Casimir energy calculations in quantum field theory in spacetimes with compactified dimensions [15].

In all the above mentioned problems, the way the dimensions are compactified (circle, parallel plates) and the relevant boundary conditions for the field (periodic, antiperiodic, Neumann, Dirichlet), may lead to a different range of summations in (3.1).

As already mentioned, the parameter M^2 usually plays the role of an (effective) mass of the theory. An approximation often employed is the large mass approximation in which one looks for an asymptotic expansion of physical quantities in inverse powers of the mass. We will now show that using the approach of section 2 this may very easily be obtained too. For the sake of generality let us consider

$$E^\beta(s) = \sum_{n=0}^{\infty} [(n+a)^\beta + M^2]^{-s} \tag{3.2}$$

where the arbitrary (positive) β leads to no additional complications and is included for that reason.

We are interested in the asymptotic expansion for large values of M^2 . Proceeding as in section 2, using the Mellin transform of the exponential, this time for $e^{-(n+a)^\beta}$, one arrives at

$$\begin{aligned} E^\beta(s) &= \frac{1}{2\pi i \Gamma(s)} \sum_{n=0}^{\infty} \int_0^\infty dt t^{s-1} e^{-M^2 t} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) (n+a)^{-\beta\alpha} t^{-\alpha} \\ &= \frac{1}{2\pi i \Gamma(s)} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) \Gamma(s-\alpha) \zeta_{\text{H}}(\beta\alpha; a) M^{2\alpha-2s}. \end{aligned} \tag{3.3}$$

Once more, the only poles that appear are of order one, located at $\alpha = 1/\beta$ (pole of the Hurwitz zeta-function) and at $\alpha = -n$, $n \in \mathbb{N}_0$ (poles of the gamma-function). The residues are easily found, leading to the asymptotic series

$$E^\beta(s) \sim \frac{\Gamma(s-1/\beta)}{\beta \Gamma(s)} \Gamma\left(\frac{1}{\beta}\right) M^{2(1/\beta-s)} + \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(s+n)}{n!} \zeta_{\text{H}}(-\beta n; a) M^{-2s-2n}. \tag{3.4}$$

For the special case $\beta = m \in \mathbb{N}$, using [20]

$$\zeta_{\text{H}}(-n; a) = -\frac{B_{n+1}(a)}{n+1}$$

with the Bernoulli polynomials $B_n(a)$, equation (3.4) may be written as

$$E^m(s) \sim \frac{\Gamma(s-1/m)}{m \Gamma(s)} \Gamma\left(\frac{1}{m}\right) M^{2(1/m-s)} - \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(s+n)}{n!} \frac{B_{mn+1}(a)}{mn+1} M^{-2s-2n}. \tag{3.5}$$

Expressions of this sort (at least in the most common case when $\beta = 2$) have been applied in the past in models of effective Lagrangians with the aim of understanding the quark

confinement problem in QCD [2]. In these theories, the heavy-mass limit for the quarks appears naturally as a good (in principle) first approximation. Different models along the same line are now fashionable again and expressions of the kind (3.5) may prove to be useful once more. On the other hand, equation (3.5) and, in particular, its partial derivatives with respect to s and M^2 are the sort of basic expressions that appear in other physical applications where compactification of spacetime plays a basic role, apart from the one already mentioned, for example Kaluza–Klein theories [23] and the computation of the vacuum energy density in compact or cylindrical universes [24].

It is rather obvious how to change the procedure in order to obtain a corresponding asymptotic expansion for the generalization of (3.2) in the form of a multi-dimensional series (see [25], where this idea is developed in detail). Whenever this approach is useful, the calculational effort (if we only keep, as here, the dominant terms of the asymptotic expansion) will not exceed the one presented in the two examples above. The corresponding results are of relevance, whenever in the context of topological mass generation [14] or the Casimir effect [15] more than one dimension is compactified.

4. Application to partition sums

In order to obtain a more detailed idea about how to use the presented techniques, let us consider the quantity

$$G(t) = \sum_{n=1}^{\infty} \ln \left(1 - e^{-t\sqrt{\lambda_n}} \right) \quad (4.1)$$

where we assume that λ_n are the eigenvalues of a positive definite elliptic differential operator L , acting on a p -dimensional Riemannian manifold \mathcal{M} with smooth boundary. In order to motivate the following considerations, let us give an example of how this quantity actually arises in concrete field-theoretic problems. Most frequently, sums like the one in equation (4.1) appear in finite-temperature quantum field theory. It was in this context, in particular, that another method involving the commutation of two (or more) series was developed. There, integral representations similar to the one in equation (2.3) have been used at some point (see for example [26]). However, making systematic use of the Mellin–Barnes type integrals from the beginning of the calculation leads to further simplification of the analysis.

For definiteness let us consider a free massive scalar field in an ultrastatic curved spacetime \mathcal{M} (possibly with boundary) [27, 28], to which we will apply the results derived in this chapter. The free energy of this system is defined by

$$F[\beta] = -\frac{1}{\beta} \ln \text{Tr} \exp[-\beta H] \quad (4.2)$$

with the Hamiltonian $H = \sum_j E_j [N_j + 1/2]$ and the inverse temperature β . Here E_j are the energy eigenvalues determined by $(-\Delta + \xi R + m^2)\psi_j = E_j^2 \psi_j$ with the Riemannian curvature R of \mathcal{M} , furthermore $-\Delta$ is the Laplace–Beltrami operator of the spatial section and m is the mass of the field. The part $(1/2) \sum_j E_j$ of the Hamiltonian H is usually called the zero-point energy of the field. The operator N_j is the number operator associated with ψ_j and the trace in (4.2) has to be taken over the Fock space of the field defined through the modes ψ_j .

A formal calculation then yields [29]

$$F[\beta] = \frac{1}{2} \sum_j E_j + \frac{1}{\beta} \sum_j \ln(1 - e^{-\beta E_j}) \quad (4.3)$$

with the divergent zero-point energy and the finite-temperature part being exactly of the form (4.1). As is seen, the limit $t \rightarrow 0$ in (4.1) corresponds to the high-temperature limit in (4.3).

Another quantity associated with (4.1) is the generalized generating function

$$Z(t) = \prod_n \left(1 - e^{-t\sqrt{\lambda_n}}\right)^{-a} \tag{4.4}$$

where a is a real number. The knowledge of its asymptotic behaviour for small t is relevant, for example, in obtaining the asymptotic state density behaviour of p -branes.

Let us now consider $G(t)$. The Mellin-transformation technique, once again, gives very easily the asymptotic expansion. First, by expanding the logarithm and using (2.2), one has

$$G(t) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\alpha \Gamma(\alpha) \zeta_R(1 + \alpha) t^{-\alpha} \zeta\left(L, \frac{\alpha}{2}\right) \tag{4.5}$$

where

$$\zeta(L, \nu) = \sum_{n=1}^{\infty} \lambda_n^{-\nu} \tag{4.6}$$

is the zeta-function associated with the sequence of eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$. In order to interchange \sum_n and the integral, one has to choose $c > p/2$.

For the evaluation of (4.5) one has to know the meromorphic structure of $\zeta(L, \nu)$. General zeta-function theory (see for example [30]) tells us that its poles of order one are located at $\nu = p/2, (p - 1)/2, \dots, 1/2; -(2l + 1)/2, l \in \mathbb{N}_0$ (we will denote the corresponding residues by R_ν and the finite part by C_ν). Thus one has the following five possible types of poles enclosed on the left of the contour:

- (i) $\alpha = p, p - 1, \dots, 1$: pole of order one due to $\zeta(L, \alpha/2)$.
- (ii) $\alpha = 0$: pole of order two due to $\Gamma(\alpha)$ and $\zeta_R(1 + \alpha)$.
- (iii) $\alpha = -1$: pole of order two due to $\Gamma(\alpha)$ and $\zeta(L, \alpha/2)$
- (iv) $\alpha = -2k, k \in \mathbb{N}$: pole of order one due to $\Gamma(\alpha)$.
- (v) $\alpha = -(2k + 1), k \in \mathbb{N}$: pole of order one due to the poles of $\Gamma(\alpha)$ and $\zeta(L, \alpha/2)$

and the zero of $\zeta_R(1 + \alpha)$.

Summing over all contributions, one arrives at

$$\begin{aligned} G(t) \sim & -2 \sum_{l=1}^p \Gamma(l) \zeta_R(1 + l) t^{-l} R_{\frac{l}{2}} - \left[\frac{1}{2} \zeta'(L, 0) - (\ln t) \zeta(L, 0) \right] \\ & - \left\{ \frac{1}{2} C_{-\frac{1}{2}} + R_{-\frac{1}{2}} [\ln(2\pi) + \psi(2) - \ln t] \right\} t \\ & - \sum_{l=1}^{\infty} \frac{t^{2l}}{(2l)!} \zeta_R(1 - 2l) \zeta(L, -l) + 2 \sum_{l=1}^{\infty} \frac{t^{2l+1}}{(2l + 1)!} \zeta'_R(-2l) R_{-(2l+1)/2} \end{aligned} \tag{4.7}$$

where the contour contributions which are exponentially damped for $t \rightarrow 0$ have not been written explicitly.

5. First physical applications: a high-temperature expansion and the asymptotic state density of p -branes

As already mentioned, a first physical application is quite immediate. We can provide the explicit form of the high-temperature expansion of a free (massive) scalar field in an

ultrastatic, curved spacetime \mathcal{M} (possibly with boundary). As briefly described, the free energy for this system is

$$F[\beta] = \frac{1}{2} \sum_j E_j + F^{(\beta)} \tag{5.1}$$

with

$$F^{(\beta)} = \frac{1}{\beta} \sum_j \ln(1 - e^{-\beta E_j}) \tag{5.2}$$

where the small t -expansion in equation (4.1) here corresponds to the high-temperature limit. By expressing the relevant information of the zeta-function $\zeta(L, \nu)$ in terms of the heat-kernel coefficients,

$$K(t) = \sum_j e^{-tE_j^2} \sim \left(\frac{1}{4\pi t}\right)^{\frac{d}{2}} \sum_{l=0,1/2,1,\dots}^{\infty} b_l t^l \tag{5.3}$$

that is

$$R_s = \frac{b_{\frac{d}{2}-s}}{(4\pi)^{\frac{d}{2}} \Gamma(s)} \quad \zeta(L, -l) = (-1)^l l! \frac{b_{\frac{d}{2}+l}}{(4\pi)^{\frac{d}{2}}} \tag{5.4}$$

the complete high-temperature expansion of the free energy may be found. Using the doubling and the reflection formula for the Γ -function, it reads:

$$F^{(\beta)} = -\frac{1}{2} P P \zeta(L, -1/2) + \frac{1}{(4\pi)^{\frac{d+1}{2}}} \left\{ -b_{\frac{d+1}{2}} \ln\left(\frac{\beta}{2\pi}\right) + \psi(2) + \frac{2\sqrt{\pi}}{\beta} b_{\frac{d}{2}} \ln \beta + P + S \right\} \tag{5.5}$$

with

$$S = - \sum_{r=1/2,1,\dots}^{\infty} b_{\frac{d+1}{2}+r} \left(\frac{\beta}{4\pi}\right)^{2r} \frac{(2r)!}{\Gamma(r+1)} \zeta_R(1+2r) \tag{5.6}$$

and

$$P = - \sum_{r=0,1/2,1,\dots}^{(d-1)/2} b_r \left(\frac{\beta}{2}\right)^{-p-1+2r} \Gamma\left(\frac{p+1}{2} - r\right) \zeta_R(p+1-2r) \tag{5.7}$$

where PP denotes the finite part of $\zeta(L, \nu)$. This is certainly the result previously found in [28].

Another application has to do with the calculation of the asymptotic state density of p -branes. Let us briefly consider it. The partition function associated with a generating function of the kind $G(t)$ may be written as

$$Z(z) = e^{-aG(z)} = \sum_n d_n e^{-nz} \tag{5.8}$$

where z is a complex variable, $z = t + iy$. The problem is to find the asymptotic behaviour of d_n for large n . This can be accomplished by making use of the asymptotic behaviour found in equation (4.7). The Cauchy integral theorem gives

$$d_n = \frac{1}{2\pi i} \oint dz e^{nz} Z(z) \tag{5.9}$$

where the contour integral consists of a small circle around the origin. For n very large, the leading contribution comes from the asymptotic behaviour of $Z(z)$ for z small. As a consequence, we may write

$$d_n \simeq \frac{A}{2\pi i} \oint dz z^B e^{zn+Cz^{-p}} \tag{5.10}$$

where

$$A = \exp\{\frac{1}{2}a\xi'(L, 0)\} \tag{5.11}$$

$$B = -a\xi(L, 0) \tag{5.12}$$

$$C = 2a\Gamma(p)\xi(1+p)R_g. \tag{5.13}$$

A straightforward calculation, based on a standard saddle-point technique, gives

$$d_n \simeq \frac{A}{\sqrt{2\pi(p+1)}} (pC)^{(2B+1)/2(p+1)} n^{-(2B+2+p)/2(p+1)} \exp\left\{\frac{p+1}{p}(pC)^{1/(p+1)} n^{p/(p+1)}\right\} \tag{5.14}$$

which generalizes Meinardus' theorem [8] (see also the recent article by Actor [9]). In the particular case of the semiclassical quantization of a p -brane, compactified on the torus T^p , this result leads to the asymptotic behaviour of the corresponding level state density for large values of the mass (see [10]).

6. Application to the asymptotics of determinants in string theory

The last application concerns the asymptotics of the determinants which appear in string theory. It is well known that the genus- g contribution to the Polyakov bosonic string partition function can be written as [31]

$$Z_g = \int (d\tau)_{WP} (\det P^+ P)^{1/2} (\det \Delta_g)^{-13} \tag{6.1}$$

where $(d\tau)_{WP}$ is the Weil–Peterson measure on the Teichmüller space, and $(\det P^+ P)$ and $\det \Delta_g$ are the scalar and ghost determinants, respectively. For our purposes, here it is sufficient to observe that the integrand can be expressed as [31]

$$I_g(\tau) = (\det P^+ P)^{1/2} (\det \Delta_g)^{-13} = e^{c(2g-2)} Z'(1)^{-13} Z(2) \tag{6.2}$$

where $Z(s)$ is the Selberg zeta-function and c is a constant. We recall that the Selberg zeta-function, for untwisted scalar fields (character $\chi(\gamma) = 1$), is defined by (for $\text{Re } s > 1$)

$$Z(s) = \prod_{\gamma} \prod_{n=0}^{\infty} (1 - e^{-(s+n)l_{\gamma}}) \tag{6.3}$$

here the product is over primitive simple closed geodesics γ on a Riemann surface and l_{γ} is the corresponding length. Furthermore, $Z(s)$ is an entire function, non-vanishing at $s = 2$ but which has a simple zero at $s = 1$ [32], corresponding to the zero mode of the scalar Laplace operator.

It is also known that physical quantities in string theory are expressed as integrals over the moduli space. The integrands are regular inside, thus, the only possible divergences have to be associated with the asymptotics near the boundary of the moduli space. It is possible to show that this boundary corresponds to the length of some geodesic tending to 0. So we are led to investigate the asymptotics for the Selberg zeta-function and its

derivatives. For the sake of simplicity, let us consider one pinching geodesic γ_1 and denote by l_1 its length. Since γ_1 and its inverse are counted as distinct primitive geodesics, we can write

$$Z(s) = R(s) \prod_{n=0}^{\infty} (1 - e^{-(s+n)l_1})^2 \tag{6.4}$$

with $R(s)$ bounded [33], and the problem reduces to finding the asymptotic behaviour of the quantities $Z(2)$ and $Z'(1)$ when l_1 goes to 0. With regard to the first quantity, this can easily be accomplished by means of the technique we have introduced. In fact, we obtain

$$\ln \frac{Z(s)}{R(s)} = -\frac{2}{2\pi i} \int_{\text{Re } z > 1} dz \Gamma(z) \zeta(1+z) \zeta_H(z; s) l_1^{-z}. \tag{6.5}$$

The simple pole at $z = 1$ and the double pole at $z = 0$ give the leading contributions, namely

$$\ln \frac{Z(s)}{R(s)} \simeq -\frac{2\zeta(2)}{l_1} + \ln l_1^{2\zeta_H(0; s)}. \tag{6.6}$$

As a result

$$Z(s) \simeq R(s) e^{-\pi^2/3l_1} l_1^{2\zeta_H(0; s)}. \tag{6.7}$$

With $\zeta_H(0; s) = 1/2 - s$, we have, in particular,

$$Z(2) \simeq R(2) e^{-\pi^2/3l_1} l_1^{-3} \tag{6.8}$$

and

$$Z(1 + \epsilon) \simeq R(1 + \epsilon) e^{-\pi^2/3l_1} l_1^{-1}. \tag{6.9}$$

Similar considerations are valid for the spinor sector, which is relevant, e.g., for superstrings. In this case, we have for $\text{Re } s > 1$ (for details we refer the reader to [31])

$$Z_1(s) = \prod_{\gamma} \prod_{n=0}^{\infty} [1 - \chi(\gamma) e^{-(s+n)l_{\gamma}}]. \tag{6.10}$$

Here the product is over primitive simple closed geodesics γ on a Riemann surface, l_{γ} is the corresponding length and the character $\chi(\gamma) = \pm 1$ depends on the spin structure. Furthermore, $Z_1(s)$ is non-vanishing at $s = 3/2$, but has a zero of order $2N$ at $s = 1/2$, N being the number of zero modes of the Dirac D operator. We also have, for the gravitino ghosts determinant,

$$\det(P_{1/2}^+ P_{1/2}) = e^{c_{1/2}} Z_1(3/2) \tag{6.11}$$

and for the square of the Dirac determinant

$$\det D^2 = e^{c_D} \frac{Z_1^{(2N)}(1/2)}{2N!}. \tag{6.12}$$

Above, $c_{1/2}$ and c_D are constants. Again the singularities are near the boundary of the moduli space and this boundary corresponds to the length of some geodesic tending to 0. Let l_1 be this length. The Ramond case ($\chi(\gamma) = 1$) is formally similar to the untwisted scalar case. So let us consider the antiperiodic case (Neveu-Schwarz), i.e. $\chi(\gamma) = -1$. We have

$$Z_1(s) = R_1(s) \prod_{n=0}^{\infty} (1 + e^{-(s+n)l_1})^2 \tag{6.13}$$

with $R_1(s)$ bounded, and the problem reduces to finding the asymptotic behaviour of the quantities $Z_1(s)$ and its derivatives at certain points, when l_1 goes to 0. Again, the Mellin technique is useful. It gives

$$\ln \frac{Z_1(s)}{R_1(s)} = \frac{1}{\pi i} \int_{\text{Re } z > 1} dz \Gamma(z)(1 - 2^{-z})\zeta(1 + z)\zeta_H(z; s)l_1^{-z}. \tag{6.14}$$

Due to the presence of the factor $1 - 2^{-z}$, we have simple poles at $z = 0, 1$. Thus, the leading contributions are

$$\ln \frac{Z_1(s)}{R_1(s)} \simeq \frac{\zeta(2)}{l_1} + 2\zeta_H(0; s) \ln 2 \tag{6.15}$$

and, as a result,

$$Z_1(s) \simeq R_1(s)e^{\pi^2/6l_1} 2^{2\zeta_H(0; s)}. \tag{6.16}$$

In particular, we have

$$Z_1(3/2) \simeq \frac{1}{4}R_1(3/2)e^{\pi^2/6l_1}. \tag{6.17}$$

The asymptotic behaviour of the derivatives at $s = 1$ and $s = 1/2$ are more difficult to investigate. A possible approach consists in starting from [32]

$$\frac{Z'(s)}{Z(s)} = \sum_{\gamma} \sum_{n=1}^{\infty} \frac{l_{\gamma}}{2 \sinh \frac{n l_{\gamma}}{2}} e^{-(s-1/2)n l_{\gamma}}. \tag{6.18}$$

Separating here the contribution of the shrinking geodesic, we have

$$\frac{Z'(s)}{Z(s)} = \sum_{n=1}^{\infty} \frac{l_1}{\sinh \frac{n l_1}{2}} e^{-(s-1/2)n l_1} + H(s). \tag{6.19}$$

Again, the Mellin technique allows us to write

$$\frac{Z'(s)}{Z(s)} = \frac{l_1}{\pi i} \int_{\text{Re } z > 2} dz \Gamma(z)\zeta(z)\zeta_H(z; s)l_1^{-z} + H(s). \tag{6.20}$$

Note that one can arrive at the same result, taking the derivative of equation (6.14) with respect to s and making use of $z\Gamma(z) = \Gamma(z + 1)$ and

$$\frac{\partial}{\partial s} \zeta_H(z; a) = -z\zeta_H(z + 1; s). \tag{6.21}$$

Thus the analogue result for the Neveu–Schwarz spinor is simply

$$\frac{Z'_1(s)}{Z_1(s)} = \frac{-l_1}{\pi i} \int_{\text{Re } z > 2} dz \Gamma(z)\zeta(z)\zeta_H(z; s)l_1^{-z}(1 - 2^{1-z}) + H_1(s). \tag{6.22}$$

In the first case, the integrand has a double pole at $z = 1$ and simple poles at $z = 0, -1, -2, \dots$. Thus, the leading contribution is

$$\frac{Z'(s)}{Z(s)} = -2[\ln l_1 + \psi(s)] + H(s) + O(l_1). \tag{6.23}$$

In the second case, we have simple poles at $z = 1, 0, -1, \dots$ and the leading term is

$$\frac{Z'_1(s)}{Z_1(s)} = -2 \ln 2 + H_1(s) + O(l_1). \tag{6.24}$$

Let us briefly discuss the bosonic case. For $\text{Re } s > 1$, there are no problems and the above result gives us the asymptotic behaviour for the logarithmic derivative of the Selberg zeta-function, when l_1 goes to 0. The delicate point is $s = 1$, where $Z(s)$ has a simple zero.

However, by using the analytical properties of the Selberg zeta-function, one may first show $R(1 + \epsilon)H(1 + \epsilon) \simeq B(1) + O(\epsilon)$, and thus we get

$$Z'(1) \simeq B(1)e^{-\pi^2/3t} l_1^{-1} \quad (6.25)$$

which gives the correct leading contribution [33]. We conclude by saying that the asymptotic behaviours found for $Z(2)$ and $Z'(1)$ lead to the celebrated double-pole theorem of Belavin and Knizhnik [34] for the quantum bosonic string.

7. Conclusions

In this article we have applied the Mellin–Barnes integral transformation to several situations of present interest in mathematical physics. We have shown, in our opinion, that this technique is best suited for the derivation of asymptotic expansions in different contexts. Compared with other approaches, it is certainly more simple and straightforward. Moreover, the expansions we have obtained may be relevant in their own right as we have seen in sections 4 to 6. Supplemented by numerical techniques and explicit results that have become available recently—adapted to the notion of asymptotic approximation—our method also constitutes a useful device in order to provide actual numbers to be contrasted with experimental measurements in fields like the determination of the Casimir energy or the study of actual physical implications of some Kaluza–Klein theories [23, 24].

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